Attribute Sampling Plans Based on Prior Distributions and Costs.

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1. A model with linear costs.

Let N and n denote lot size and sample size, and let X and x denote number of defectives in the lot and the sample, respectively. The acceptance number is denoted by c.

Let the costs be

$$nS_1 + xS_2 + (N-n)A_1 + (X-x)A_2 \qquad \text{for } x \leq c \tag{1}$$

and

$$nS_1 + xS_2 + (N-n)R_1 + (X-x)R_2 \qquad \text{for } x > c. \tag{2}$$

The interpretation of the six cost parameters depends on the kind of inspection envisaged, i.e. whether inspection is a consumer's receiving inspection, a producer's inspection of finished goods, or "internal inspection" by delivery of goods from one department to another within the same firm. The cost parameters may have quite different values when considered exclusively from a producer's or a consumer's point of view because certain costs are borne primarily by one of the parties involved. The values of the cost parameters also depend on whether the inspection is rectifying or non-rectifying, destructive or non-destructive. In the following the two cost expressions are discussed and a few examples of interpretation are given.

Costs associated with the sample, $nS_1 + xS_2$, for brevity called "costs of sampling inspection", consist of two parts: one part, nS_1, proportional to the number of items in the sample so that S_1 includes sampling and testing costs per item, and another part, xS_2, proportional to the number of defectives in the sample, i.e. S_2 denotes additional costs for an inspected defective item. If defective items found in the sample are repaired, say, then S_2 includes repair costs per item.

"Costs of acceptance" are similarly composed of a part, $(N-n)A_1$, proportional to the number of items in the remainder of the lot, and another part, $(X-x)A_2$, proportional to the number of defective items accepted. Whereas A_1 usually will be zero or negligible, A_2 will often be considerable. If accepted items are used

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Invited paper to the EOQC Conference, Stockholm, June 1966.

Prepared with the partial support of the Office of Naval Research (Nonr-N62558-3073).

Institute of Mathematical Statistics, University of Copenhagen.
Technical Report No. 11 to the Office of Naval Research
prepared under Contract Nonr-N62558-3073 (NR 042-225).

as parts in an assemply operation, say, A_2 may include the manufacturing costs (or the price) of an item, the costs of handling the defective item in assembling and disassembling, and the damage done to other parts used in the assembly. In case of inspection of finished goods A_2 may include costs of repair, service and guarantees plus loss of good-will.

"Costs of rejection" consist of a part, $(N-n)R_1$ proportional to the number of items in the remainder of the lot, and another part, $(X-x)R_2$, proportional to the number of defective items rejected. Rejection is here taken in a broad sense meaning only that the lot cannot be accepted according to the sampling plan used. Rejection may therefore lead to sorting, price reduction, scrapping, or salvaging. If rejection means sorting, say, then R_1 includes sorting costs per item and R_2 denotes additional costs for defective items found, for example costs of repair or replacement.

It is obvious that from a practical point of view it will in general be easiest to obtain information on the values of the cost parameters in the case of "internal inspection".

Let $f_N(X)$ denote the (prior) distribution of X, i.e. the distribution of lot quality. It is assumed that this distribution is a mixed binomial distribution, i.e.

$$f_N(X) = \int_0^1 \binom{N}{X} p^X q^{N-X} dW(p) \tag{3}$$

where W(p) denotes a cumulative distribution function (independent of N).

Drawing a sample without replacement from each lot (hypergeometric sampling) and computing the average costs we find

$$K(N,n,c) = \int_0^1 K(N,n,c,p)dW(p) \tag{4}$$

where

$$K(N,n,c,p) = n(S_1+ S_2p) + (N-n)((A_1+ A_2p)P(p) + (R_1+ R_2p)Q(p)), \tag{5}$$

and

$$P(p) = B(c,n,p) = \sum_{x=0}^{c} \binom{n}{x} p^x q^{n-x}, \quad Q(p) = 1-P(p). \tag{6}$$

For convenience the frequency function corresponding to W(p) will be called the distribution of the process average or the distribution of p as distinct from $f_N(X)$ which gives the distribution of X/N, i.e. the distribution of lot quality.

Starting from (5) we introduce the three cost functions

$$k_s(p) = S_1+ S_2p, \quad k_a(p) = A_1+ A_2p, \quad k_r(p) = R_1+ R_2p, \tag{7}$$

the (economic) break-even quality,

$$p_r= (R_1- A_1)/(A_2-R_2), \; 0 < p_r < 1, \text{ and}$$

$$k_m(p) = k_a(p) \text{ for } p \leq p_r \text{ and } k_r(p) \text{ for } p > p_r. \tag{8}$$

The function $k_m(p)$ gives the unavoidable (minimum) costs, i.e. the costs corresponding to the situation where perfect knowledge of quality exists without costs of information and all lots are classified correctly on the basis of the corresponding process average, viz. accepted for $p \leq p_r$ and rejected for $p > p_r$.

Averages over the prior distribution are denoted by k_s, k_a, etc., and costs per item are denoted by k, and costs per lot by the corresponding K, i.e. $K = Nk$. The average costs for the three cases without sampling inspection, i.e. the cases where (a) all lots are classified correctly, (b) all lots are accepted, and (c) all lots are rejected, then become k_m, k_a, and k_r respectively. These cases are useful "reference cases" since sampling inspection is justified only if $k-k_m < \min\{k_a - k_m, k_r - k_m\}$, where $k = K(N,n,c)/N$.

Case (a) will usually be considered as the basic reference case and average costs for other cases will therefore be reduced by k_m, since k_m represents the average fixed costs per item which will be incurred irrespective of the decision made. The cost differences

$$k_a - k_m = \int_{p_r}^{1} (k_a(p)-k_r(p))dW(p) \quad \text{and} \quad k_r - k_m = \int_{0}^{p_r} (k_r(p)-k_a(p))dW(p)$$

represent average decision losses in case (b) and (c) respectively, and $k_s - k_m$ represents the average "loss" by inspection.

From (4) and (5) we find

$$K-K_m = n(k_s-k_m)+(N-n)(A_2-R_2)\{\int_0^{p_r}(p_r-p)Q(p)dW(p) + \int_{p_r}^{1}(p-p_r)P(p)dW(p)\}, \tag{9}$$

the two terms giving the average costs of sampling inspection and the average decision losses, respectively.

Putting $R(N,n,c) = \{K(N,n,c)-K_m\}/(k_s - k_m)$, we find

$$R = n + \frac{N-n}{p_s-p_m}\{ \int_0^{p_r}(p_r-p)Q(p)dW(p) + \int_{p_r}^{1}(p-p_r)P(p)dW(p)\}, \tag{10}$$

the two terms again giving the costs of sampling inspection and the average decision losses, respectively, but here using the average costs of sampling inspection (minus k_m) per item in the sample as economic unit.

2. Results for double binomial prior distributions.

The simplest theory of sampling inspection is based on the assumption that lots submitted for inspection originate from one of two possible quality levels, p_1 and p_2, $p_1 < p_2$, and that the process average p_1 occurs with probability w_1,

p_2 consequently with probability $w_2 = 1-w_1$. The standardized cost function (10) then takes the form

$$R = n + (N-n)(\gamma_1 Q(p_1) + \gamma_2 P(p_2)) \qquad (11)$$

where $\gamma_i = |p_i - p_r| w_i/(p_s - p_m) = |k_a(p_i) - k_r(p_i)| w_i/(k_s - k_m)$, $i = 1,2$, and $p_m = p_1 w_1 + p_r w_2$, i.e. R depends on four parameters only, viz. $p_1, p_2, \gamma_1, \gamma_2$.

Since $P(p) = B(c,n,p)$ denotes the operating characteristic, $Q(p_1)$ and $P(p_2)$ equal <u>the producer's and the consumer's risks,</u> respectively.

We shall discuss ten systems of sampling plans defined as follows:

(1). <u>Bayesian plans,</u> i.e. plans minimizing R.

<u>Restricted Bayesian plans</u>, i.e. plans minimizing R under some suitably chosen restriction on the operating characteristic, viz.

(2). Min R for $Q(p_1) = \alpha$ <u>or</u> $P(p_2) = \beta$.

(3). Min R for $P(p_0) = 1/2$ where $p_0 = (\log \frac{q_1}{q_2})/(\log \frac{p_2 q_1}{p_1 q_2})$.

(4). Min R for $Q(p_1) = \alpha/N$ <u>or</u> $P(p_2) = \beta/N$.

(5). Min R for $P(p_2)/Q(p_1) = \rho$.

<u>Plans defined by two risks,</u> viz.

(6). $Q(p_1) = \alpha/N$ <u>and</u> $P(p_2) = \beta/N$.

(7). $Q(p_1) = \alpha$ <u>and</u> $P(p_2) = \beta/N$ (or $P(p_2) = \beta$ <u>and</u> $Q(p_1) = \alpha/N$).

(8). $P(p_0) = 1/2$ <u>and</u> $Q(p_1) = \alpha/N$ (or $P(p_2) = \beta/N$).

(9). $Q(p_1) = \alpha$ <u>and</u> $P(p_2) = \beta$.

Finally we consider <u>percentage inspection</u> defined as

(10). $n = \mu N$ and $c = p_0 n$.

In all these definitions α, β, ρ, and μ represent suitably chosen positive constants which may be different from case to case.

For each system of sampling plans it has been shown in [1] how the exact solution may be obtained and, since this solution is an implicit one, an explicit solution is given as an asymptotic expansion for $N \to \infty$. Tables have been provided in [2] and [3].

In this section we shall assume that the quality distribution of submitted lots is a double binomial and compare the costs of the various systems under this assumption. As a measure of efficiency we use the ratio $e(N,n,c) = R_0(N)/R(N,n,c)$, where $R_0(N)$ denotes the costs of the optimum (Bayesian) plan and $R(N,n,c)$ denotes the costs of the plan in question.

The systems defined by (1) - (8) fall into two classes depending on whether <u>both risks are $O(N^{-1})$</u> or <u>one of the risks is constant and the other is $O(N^{-1})$.</u>

<u>The first class</u> contains systems (1), (3), (4), (5), (6), and (8). Asymptotically the relation between acceptance number and sample size has the form $c = p_0 n + a_2 + a_4 n^{-1} + \underline{O}(n^{-2})$, and $\ln N = \varphi_0 n + \frac{1}{2}\ln n + \kappa_1 + \kappa_2 n^{-1} + \underline{O}(n^{-2})$, which by inversion determines n as a function of N, and $R = n + \delta_1 + \delta_2 n^{-1} + \underline{O}(n^{-2})$, the constants p_0 and φ_0 being the same in all cases (depending on (p_1, p_2) only), whereas the remaining constants are found as functions of the parameters in the model and the restriction. This means that $n = \underline{O}(\ln N)$ and that the average decision loss, $(N-n)(\gamma_1 Q(p_1) + \gamma_2 P(p_2))$, tends to a constant (because $Q(p_1)$ and $P(p_2)$ are $\underline{O}(N^{-1})$).

<u>The second class</u> consists of systems (2) and (7). Asymptotically the relation between acceptance number and sample size has the form $c = p_j n + \sum_{i=1}^{4} a_i n^{1-i/2} + \underline{O}(n^{-3/2})$, p_j representing the quality level having a constant risk, $\ln N = \varphi_j n + \frac{1}{2}\ln n + \sum_{i=1}^{4} \kappa_i n^{1-i/2} + \underline{O}(n^{-3/2})$, which determines n as a function of N, and $R = \delta N + (1-\delta)n + \delta_1 + \underline{O}(n^{-1/2})$, the constants φ_j and δ being the same in the two cases. Because of the constant risk all relations are considerably more complicated than for the first class and R becomes $\underline{O}(N)$ instead of $\underline{O}(\ln N)$. <u>For large lots it is therefore not advisable to use a system with a fixed consumer's or producer's risk</u> and correspondingly high costs as compared with a system having decreasing risks.

The system with <u>both risks fixed</u> and the system with <u>percentage inspection</u> both lead to $R = \underline{O}(N)$ and <u>asymptotically they have the same costs for</u> $\mu = \gamma_1\alpha + \gamma_2\beta$. The system with fixed risks uses a fixed sample size so that the decision loss becomes of order N, whereas percentage inspection has $n = \underline{O}(N)$ and a decision loss of order e^{-N}.

The restricted Bayesian plans with both risks decreasing and the corresponding plans based on two risks have an economic efficiency tending to 1 for $N \to \infty$ as compared to the Bayesian plans. (The efficiency of plans having at least one risk fixed tends to zero). This result means that wrong values of the weights of the prior distribution and wrong values of the cost parameters have a secondary influence on the efficiency which tends to 1 if only (p_1, p_2) are correct. If wrong values of (p_1, p_2), (p_1^*, p_2^*) say, are used for finding the plans then the efficiency tends to e, $0 < e < 1$, if and only if $p_1 < p_1^* < p_2^* < p_2$, otherwise the efficiency tends to 0.

As an example consider a case with $p_1 = 0.01$, $w_1 = 0.85$, and $p_2 = 0.05$, $w_2 = 0.15$. Let the costs of sampling inspection be 0.40 (economic units) per item in the sample, i.e. $S_1 = 0.40$ and $S_2 = 0$, the costs of rejection 0.30 per item in the remainder, i.e. $R_1 = 0.30$ and $R_2 = 0$, and the costs of acceptance per defective item 10.00, i.e. $A_1 = 0$ and $A_2 = 10.00$. It follows that $p_r = 0.03$, $\gamma_1 = 0.6296$, and $\gamma_2 = 0.1111$.

In the table we have compared plans from 9 systems defined as follows:

(1). Bayes. Plans minimizing $R(N,n,c)$.

(2). IQL. Min R for $P(p_0) = 1/2$, $p_0 = 0.0250$.

(3). LTPD. Min R for $P(p_2) = 0.10$.

(4). AQL. Min R for $Q(p_1) = 0.05$.

(5). Fixed risk. $Q(p_1) = 0.05$ and $P(p_2) = 0.10$.

(6). Percentage inspection. $\mu = 0.05\gamma_1 + 0.10\gamma_2 = 0.04259$.

(7). Dodge. The AQL system with 5% consumer's risk proposed by Dodge in [6] with AQL = p_1.

(8). Mil-Std. Military-Standard 105D with AQL = p_1.

(9). Minimax regret plans obtained by minimizing $\max_p \{K(N,n,c,p) - Nk_m(p)\}$.

For each of 7 lot sizes the plans and the corresponding costs have been found and the efficiency has been computed.

3. Results for continuous prior distributions of process average.

If the process average varies at random according to a continuous distribution with density $w(p)$ we get optimum (Bayesian) plans with properties quite different from those described in section 2. The asymptotic results, which have been derived in [4], are the following: $c = p_r n + a + \underline{O}(n^{-1})$, $n = \lambda_1\sqrt{N} + \lambda_2 + o(1)$, and $R = 2n + o(1)$, where a, λ_1, and λ_2 depend on $p_r, w(p_r)$, and the first two derivatives of $w(p)$ for $p = p_r$.

Furthermore, the asymptotic efficiency of a plan (N,n_1,c_1) in relation to the optimum plan (N,n_0,c_0) is

$$e(N,n_1,c_1) = 2\Big/\left(\frac{n_1}{n_0} + \frac{n_0}{n_1}\right) \tag{12}$$

if both plans use the right relation between sample size and acceptance number, otherwise the efficiency tends to zero.

4. The minimax regret solution.

If the prior distribution is unknown one may use the minimax regret method to derive a sampling plan, see [7]. The main asymptotic results are the following: $c = p_r n + a + \underline{O}(n^{-1})$ and $n = \lambda_1 N^{2/3} + \lambda_2 N^{1/3} + \underline{O}(1)$, where $\underline{a}$ depends on p_r, λ_1 and λ_2 on p_r and the cost constants.

Comparisons of sampling plans.

	Bayes			IQL			LTPD			AQL			Fixed Risk			Percentage Insp			Dodge			Mil - Std			Minimax		
N	n	c	R	n	c	100e	n	c	100e	n	c	100e	n	c	100e	n	c	100e	n	c	100e	n	c	100e	n	c	100e
100	accept		11	27	0	28	45	0	19	5	0	69	-	-	-	4	0	74	40	1	25	13	0	45	21	0	33
300	accept		33	27	0	44	77	1	32	5	0	84	133	3	24	13	0	65	40	1	54	50	1	46	50	1	46
1000	accept		111	67	1	66	105	2	67	35	1	97	133	3	67	43	1	90	100	2	71	80	2	85	87	2	80
3000	105	3	217	187	4	80	158	4	95	82	2	89	133	3	88	128	3	91	150	3	73	125	3	93	187	5	92
10000	220	6	344	307	7	83	209	6	98	137	3	64	133	3	66	426	11	77	250	5	66	200	5	94	452	13	73
30000	340	9	465	427	10	89	234	7	77	262	5	38	133	3	36	1278	32	36	400	7	35	315	7	74	952	28	49
100000	495	13	601	547	13	91	282	9	42	329	6	17	133	3	15	4259	106	14	650	10	13	510	10	45	2119	63	28

N	Bayes $100Q_1$	Bayes $100P_2$	IQL $100Q_1$	IQL $100P_2$	LTPD $100Q_1$	LTPD $100P_2$	AQL $100Q_1$	AQL $100P_2$	Fixed Risk $100Q_1$	Fixed Risk $100P_2$	Percentage Insp $100Q_1$	Percentage Insp $100P_2$	Dodge $100Q_1$	Dodge $100P_2$	Mil - Std $100Q_1$	Mil - Std $100P_2$	Minimax $100Q_1$	Minimax $100P_2$
100	0.0	100.0	23.8	25.0	36.4	9.9	4.9	77.4	-	-	3.9	81.5	6.1	39.9	12.2	51.3	19.0	34.1
300	0.0	100.0	23.8	25.0	18.0	9.7	4.9	77.4	4.5	9.6	12.2	51.3	6.1	39.9	8.9	27.9	8.9	27.9
1000	0.0	100.0	14.5	14.6	8.9	9.9	4.8	47.2	4.5	9.6	6.9	36.0	7.9	11.8	4.7	23.1	5.7	18.4
3000	2.2	22.4	4.1	4.1	2.2	10.0	5.0	21.6	4.5	9.6	4.0	11.3	6.5	5.5	3.7	12.4	1.2	9.0
10000	0.71	7.3	1.3	1.3	0.54	9.8	5.0	8.4	4.5	9.6	0.15	0.96	4.1	1.3	1.6	6.2	0.02	1.9
30000	0.26	2.3	0.44	0.44	0.27	9.8	5.0	0.88	4.5	9.6	$0.0^{3}4$	$0.0^{3}5$	5.0	0.062	1.5	1.0	0.00	0.1
100000	0.059	0.63	0.15	0.15	0.064	9.9	4.9	0.25	4.5	9.6	$0.0^{14}6$	$0.0^{3}3$	6.6	$0.0^{3}25$	1.3	0.046	0.00	0.0

$Q_1 = Q(p_1) = 1-B(c,n,p_1)$, $P_1 = 0.01$, and $P_2 = P(p_2) = B(c,n,p_2)$, $P_2 = 0.05$.

5. A more general model.

The model above is based on the assumptions that costs are linear in p, and that the distribution of the number of defectives in the sample (for given p) is binomial.

In the present section we shall state the asymptotic results obtained in a recent paper [5] concerning a much more general model.

Consider a problem with two decisions only, acceptance and rejection say, where acceptance is preferred if the unknown parameter θ is less than or equal to a specified break-even value θ_0, and rejection is preferred for $\theta > \theta_0$. We assume that the statistic $t = t(x_1,\dots,x_n)$ is sufficient for θ, and that the decision rule has the form: "Accept if $t \leq t^0$ and reject otherwise." The problem is to determine the optimum n and t^0.

We shall introduce several functions of θ such as $w(\theta)$, $\alpha(\theta)$, etc. and to shorten the notation we write $w(\theta_0) = w$, $w'(\theta_0) = w'$, etc.

The model is based on the following assumptions:

(1). The density $w(\theta)$ of the prior distribution of θ is twice differentiable in an open interval about the break-even value θ_0 and $w > 0$.

(2). The loss function $\underline{l}(\theta)$, i.e. the loss of a wrong decision, may be written as

$$\underline{l}(\theta) = \begin{cases} \underline{l}_1(\theta_0-\theta)^{\nu_1}(1+\underline{l}_{11}(\theta-\theta_0)+\dots) & \text{for } \theta \leq \theta_0 \\ \underline{l}_2(\theta-\theta_0)^{\nu_2}(1+\underline{l}_{21}(\theta-\theta_0)+\dots) & \text{for } \theta > \theta_0 \end{cases} \tag{13}$$

where (ν_1,ν_2) are non-negative and $(\underline{l}_1,\underline{l}_2)$ are positive constants.

(3). Conditional on θ, the statistic t has mean θ, variance $\alpha^2(\theta)/n$, and coefficient of skewness $\beta(\theta)/\sqrt{n}$, and the operating characteristic $P(\theta)$ of the decision rule, i.e. $\Pr\{t \leq t^0|\theta\}$, may be written as an Edgeworth expansion

$$P(\theta) = \Phi(x)-\beta(\theta)(x^2-1)\varphi(x)/6\sqrt{n} + \underline{O}(n^{-1}), \qquad x = (t^0-\theta)\sqrt{n}/\alpha(\theta) \tag{14}$$

where φ and Φ denote the density and the cumulative distribution function for the standardized normal distribution. The functions $\alpha(\theta)$ and $\beta(\theta)$ are assumed to be twice differentiable.

(4). The (expected) sampling costs are $r_s(n) = kn^s(1+\underline{O}(n^{-1}))$.

From these assumptions we find the average regret $r(n,t^0)$ as

$$r(n,t^0) = r_s(n)+r_1(n,t^0), \tag{15}$$

where the average decision loss is

$$r_{\underline{1}}(n,t^0) = \int_{-\infty}^{\theta_0} \underline{1}(\theta)Q(\theta)w(\theta)d\theta + \int_{\theta_0}^{\infty} \underline{1}(\theta)P(\theta)w(\theta)d\theta , \tag{16}$$

with $Q(\theta) = 1-P(\theta)$.

The Bayesian sampling plan (n_0,t_0) is found by minimizing $r(n,t^0)$. Because of the continuity of $\underline{1}(\theta)$, $P(\theta)$, and $w(\theta)$ in the neighbourhood of θ_0 and the decreasing standard deviation of t, the asymptotic solution depends only on the properties of the functions involved at θ_0.

It is obvious that we might have chosen one of the costs constants, k for example, as the economic unit, i.e. the solution will depend on the ratios $\underline{1}_1/k$ and $\underline{1}_2/k$ only. In deriving the asymptotic results we assume that the two ratios tend to infinity, and that $\underline{1}_2/\underline{1}_1$ is constant. We shall treat n and t^0 as continuous variables.

From (15) it follows that the optimum values of $(n,t^0)$are determined from the equations $\partial r_{\underline{1}}/\partial t^0 = 0$ and $-\partial r_{\underline{1}}/\partial n = \partial r_s/\partial n$. For any n the optimum value of t^0, i.e. the value minimizing $r(n,t^0)$, may be found by solving the first of these equations. The solution will be denoted $t_0(n)$. Inserting $t^0 = t_0(n)$ into the second equation we get an equation in n alone which gives $n = n_0$ and finally $t_0 = t_0(n_0)$. The functions $r_{\underline{1}}(n,t_0(n))$ and $r(n,t_0(n))$ will be denoted $r_{\underline{1}}(n)$ and $r(n)$, respectively.

Comparing (10) with (15) and (16) we find that (asymptotically) the model simplifies to the one considered above, if the cost functions are chosen as $r_s(n) = (p_s-p_m)n$ and $\underline{1}(p) = N|p-p_r|$, and if furthermore $\alpha^2(p) = pq$ and $\beta(p) = (q-p)/\sqrt{pq}$, since (14) then gives the Edgeworth expansion for the cumulative binomial distribution.

It is, however, clear that the present model also covers cases with non-linear cost functions, and operating characteristics different from the binomial, for example, the Poisson and the normal.

As auxiliary functions we introduce the incomplete normal moment of order ν

$$m_\nu(z) = \int_0^\infty t^\nu \varphi(t+z)dt = \int_z^\infty (t-z)^\nu \varphi(t)dt, \qquad \nu \geq 0, \tag{17}$$

and

$$\underline{1}_\nu(z) = \underline{1}_1 m_\nu(z) + \underline{1}_2 m_\nu(-z). \tag{18}$$

As a measure of efficiency of a non-optimum plan (n,t^0) we shall use the ratio

$$e(n,t^0) = r(n_0,t_0)/r(n,t^0).$$

This measure of efficiency lies between 0 and 1, it is invariant to the choice of origin and scale of costs (or utilities), and it expresses directly what costs could be reduced to (by using the optimum sampling plan) as a fraction of the costs for the non-optimum plan.

If $t^0 = \theta_1 + o(1)$, $\theta_1 \neq \theta_0$, it is easy to show that $e \to 0$ as $n \to \infty$. The break-even value θ_0 is therefore the most important parameter.

We shall in particular study the efficiency first under the assumption that $t_0(n)$ is known, so that the efficiency becomes $e(n) = r(n_0)/r(n)$, and next under the assumption that only θ_0 is known, so that $t^0 = \theta_0$ and $e(n,\theta_0) = r(n_0,t_0)/r(n,\theta_0)$.

In stating the results obtained it is essential to distinguish between two cases, $\nu_2 > \nu_1$ say, and $\nu_2 = \nu_1 = \nu$.

<u>Results for</u> $\nu_2 = \nu_1 = \nu$.

The optimum relationship between t^0 and n is given by

$$t_0 = \theta_0 + a\alpha/\sqrt{n} + b\alpha/n + \underline{O}(n^{-3/2}), \tag{19}$$

where a and b are determined from the equations

$$\underline{1}_1 m_\nu(a) = \underline{1}_2 m_\nu(-a) \tag{20}$$

and

$$b = -\alpha w'/w - (\nu+1)\alpha' + (a^2+\nu-1)\beta/6 - \alpha\,\{\underline{1}_{11}\underline{1}_1 m_{\nu+1}(a) + \underline{1}_{21}\underline{1}_2 m_{\nu+1}(-a)\}/\underline{1}_{\nu+1}(a). \tag{21}$$

The optimum sample size is

$$n_0 = \gamma^{2/(2s+\nu+1)}\left(1+2\delta\gamma^{-1/(2s+\nu+1)}/(2s+\nu+1) + \underline{O}(\gamma^{-2/(2s+\nu+1)})\right) \tag{22}$$

where $\gamma = w\alpha^{\nu+1}\,\underline{1}_{\nu+1}(a)/2sk$ and

$$\delta = a(\alpha w'/w + (\nu+1)\beta/6) + \alpha\{-\underline{1}_{11}\underline{1}_1 m_{\nu+2}(a) + \underline{1}_{21}\underline{1}_2 m_{\nu+2}(-a)\}/\underline{1}_{\nu+1}(a).$$

The average decision loss for $t^0 = t_0(n)$ is

$$r_{\underline{1}}(n) = \frac{w\underline{1}_{\nu+1}(a)}{\nu+1}\left(\frac{\alpha}{\sqrt{n}}\right)^{\nu+1}\left(1 + \frac{(\nu+1)\delta}{(\nu+2)\sqrt{n}} + \underline{O}(\tfrac{1}{n})\right), \tag{23}$$

and the minimum regret becomes

$$r(n_0) = r_s(n_0)\,\frac{2s+\nu+1}{\nu+1}\left(1 - \frac{2s\delta}{(\nu+2)(2s+\nu+1)\sqrt{n_0}} + \underline{O}(\tfrac{1}{n_0})\right). \tag{24}$$

For the efficiency we get

$$\frac{1}{e(n)} \sim \frac{\nu+1}{2s+\nu+1}\left(\frac{n}{n_0}\right)^s + \frac{2s}{2s+\nu+1}\left(\frac{n_0}{n}\right)^{(\nu+1)/2} \tag{25}$$

and

$$\frac{1}{e(n,\theta_0)} \sim \frac{\nu+1}{2s+\nu+1}\left(\frac{n}{n_0}\right)^s + \frac{2s}{2s+\nu+1}\left(\frac{n_0}{n}\right)^{(\nu+1)/2} \frac{\underline{1}_{\nu+1}(0)}{\underline{1}_{\nu+1}(a)}\ , \tag{26}$$

i.e. the second member of (25) has been changed by a constant factor.

It follows that $t_0 \to \theta_0$, the difference $t_0 - \theta_0$ being of order $1/\sqrt{n}$ if $\underline{1}_1 \neq \underline{1}_2$ (non-symmetric loss function), and of order $1/n$ for $\underline{1}_1 = \underline{1}_2$ because then $a = 0$ from (20).

If $\underline{1}_1$ and $\underline{1}_2$ are proportional to lot size N then the optimum sample size n_0 becomes proportional to $N^{2/(2s+\nu+1)}$ according to (22). By varying $s \geq 1$ and $\nu \geq 0$ we may get all powers of N less than or equal to 2/3. For $\nu = s = 1$ we get $N^{1/2}$.

For the optimum plan the ratio of the average decision loss to the sampling costs tends to $2s/(\nu+1)$, cf. (24).

From (25) it follows that the efficiency of a plan based on the right relationship between t^0 and n will be high, even if n deviates considerably from n_0. For $\nu = s = 1$ we get (12).

If θ_0 is known we may get a rather high efficiency by using $t^0 = \theta_0$ even if n deviates from n_0, see (26).

If n has been determined from a wrongly chosen s we get $n \propto n_0^\delta$, $\delta \neq 1$, or if a discrete prior distribution has been used instead of the correct continuous one we get $n \propto \ln n_0$, and in both cases $\lim e(n) = 0$.

For the simple symmetric loss function $\underline{1}(\theta) = \underline{1}_0|\theta - \theta_0|^\nu$ one more term of the above expansions may be found in [5].

<u>Results for</u> $\nu_2 > \nu_1$.

The optimum relationship between t^0 and n is given by

$$t_0 = \theta_0 + \alpha m/\sqrt{n} - \alpha((\nu_1+\nu_2+1)\ln m + \ln\lambda)/(m\sqrt{n}) + o(1/m^2\sqrt{n}) \tag{27}$$

where $m = \sqrt{(\nu_2 - \nu_1)\ln n}$ and $\lambda = (\sqrt{2\pi}\,\alpha^{\nu_2-\nu_1}\,\underline{1}_2)/(\underline{1}_1\Gamma(\nu_1 + 1))$.

The optimum sample size is

$$n_0 = \gamma_2^{2/(2s+\nu_2+1)}\{(2\ln\gamma_2)/(2s+\nu_2+1)\}^{(\nu_2+1)/(2s+\nu_2+1)}(1+o(1)), \tag{28}$$

where $\gamma_2 = w\alpha^{\nu_2+1}(\nu_2-\nu_1)^{(\nu_2+1)/2}\underline{1}_2/2sk$.

The average decision loss for $t^0 = t_0(n)$ is

$$r_{\underline{1}}(n) = (w\underline{l}_2/(\nu_2+1))(\alpha m/\sqrt{n})^{\nu_2+1}(1+o(m^{-1})) \tag{29}$$

and the minimum regret becomes

$$r(n_0) = r_s(n_0)(2s+\nu_2+1)/(\nu_2+1)(1+o(m_0^{-1})). \tag{30}$$

For the efficiency we get

$$\frac{1}{e(n)} \sim \frac{\nu_2+1}{2s+\nu_2+1}\left(\frac{n}{n_0}\right)^s + \frac{2s}{2s+\nu_2+1}\left(\frac{n_0 \ln n}{n \ln n_0}\right)^{(\nu_2+1)/2} \tag{31}$$

and

$$\frac{1}{e(n,\theta_0)} \sim \frac{\nu_2+1}{2s+\nu_2+1}\left(\frac{n}{n_0}\right)^s + \frac{2s}{2s+\nu_2+1}\left(\frac{n_0 \ln n}{n \ln n_0}\right)^{(\nu_2+1)/2}\left[\frac{n^{(\nu_2-\nu_1)/2}}{m^{\nu_2+1}}\right]\frac{\underline{1}_1 m_{\nu_1+1}(0)(\nu_2+1)}{\underline{1}_2\alpha^{\nu_2-\nu_1}(\nu_1+1)} \tag{32}$$

so that $\lim e(n,\theta_0) = 0$ even if n is proportional to n_0.

For $\nu_2 > \nu_1$ it is more serious to reject a lot which should have been accepted than to accept a lot which should have been rejected. (It should be remembered that for large n it is only losses in the neighbourhood of θ_0 that matter). This is the reason for the result that t_0 tends to θ_0 from above, the difference $t_0 - \theta_0$ being of order $\sqrt{(\ln n)/n}$, so that the rate of convergence is considerably slower than for $\nu_2 = \nu_1$.

As a consequence of this result the average decision loss consists essentially of the loss from wrongly accepting bad lots. For the optimum plan the ratio of the average decision loss to the sampling costs tends to $2s/(\nu_2+1)$, see (30).

As a further consequence it will be seen from (32) that it is not satisfactory to put $t^0 = \theta_0$ because such a plan will have an efficiency tending to zero for $n \to \infty$.

The results above depend essentially on the assumption that the density of the prior distribution is positive and differentiable in the neighbourhood of θ_0, and furthermore on the assumption that $l(\theta_0) = 0$ and $l(\theta) > 0$ for $\theta \neq \theta_0$.

If the prior distribution is <u>discrete</u>, for example a two-point distribution given by $Pr\{\theta=\theta_i\} = w_i$, $i = 1,2$, then we get similar results as in section 2, i.e. $t_0 \to \theta^*$ where θ^* depends on θ_1 and θ_2 only, n_0 becomes proportional to the logarithm of a linear combination of $\underline{l}(\theta_1)$ and $\underline{l}(\theta_2)$, and the ratio of the average decision loss to the sampling costs will be of order n_0^{-1}.

If $\underline{l}(\theta) = 0$ for $\theta_1 \leq \theta \leq \theta_2$, i.e. there exists an <u>indifference zone</u> instead of a break-even point, then we get similar results as for a discrete prior distribution.

Considering restricted Bayesian plans it is clear that any restriction of the form $P(\theta_1) = P$, say, where $\theta_1 \neq \theta_0$ and P is a given number, will lead to $t^0 \longrightarrow \theta_1$ and therefore to an efficiency tending to zero. Restrictions should therefore be of the form $P(\theta_0) = P$ or $P(\theta_2)/Q(\theta_1) = \rho$, where ρ is a constant and $\theta_1 < \theta_0 < \theta_2$.

6. Discussion.

Bayesian sampling plans should be used if the prior distribution is known and stable, if the sampling costs and the decision losses are known, and if the purpose is to minimize the sum of the average sampling costs and decision losses for a series of lots.

The above results show how the Bayesian plans depend on the assumptions. We have found, for example, how the optimum sample size depends on the main parameters in the model. Further investigations are needed to evaluate these results. It is, however, fortunate that the formulas for the efficiency of non-optimum plans show a high degree of insensitivity to deviations from the optimum sample size if only the break-even point is known.

In practice a system of sampling plans is often required to serve several porposes - besides being easy to administer. In particular we shall here mention (a) that the system should protect the consumer against the consequences of deterioration of the prior distribution, (b) that the system should work as an incentive for the producer to produce better quality or at least to keep to the quality agreed upon, and (c) that average costs should be minimized. So far, however, there has not been developed a theory taking all these aspects into account.

The purposes (a) and (b) above may be obtained by introducing restrictions on the operating characteristic, and by alternating between normal and tightened inspection. An approximation to a solution may therefore be obtained by using a restricted Bayesian plan with decreasing producer's and consumer's risks combined with a feed-back mechanism which induces shifts between normal and tightened inspection according to changes in the prior distribution.

It would be very useful for the direction of further research in this area if inspection departments would publish their experiences with respect to costs and prior distributions. Great masses of data must exist which could help to solve problems regarding the form and stability of prior distributions.

References.

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